A symmetry trip from Caldirola to Bateman damped systems

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Abstract

For the Caldirola-Kanai system, describing a quantum damped harmonic oscillator, a couple of constant-of-motion operators generating the Heisenberg algebra can be found. The inclusion of the standard time evolution symmetry in this algebra for damped systems, in a unitary manner, requires a non-trivial extension of this basic algebra and hence the physical system itself. Surprisingly, this extension leads directly to the so-called Bateman's dual system, which now includes a new particle acting as an energy reservoir. The group of symmetries of the dual system is presented, as well as a quantization that implies, in particular, a first-order Schrödinger equation. The usual second-order equation and the inclusion of the original Caldirola-Kanai model in Bateman's system are also discussed.

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1 Introduction

The interest in dissipative systems at the quantum level has remained constant since the early days of Quantum Mechanics. The difficulties in describing damping, which intuitively could be understood as a mesoscopic property, within the fundamental quantum framework, have motivated a huge amount of papers.

Applications of quantum dissipation abound. For example, in quantum optics, where the quantum theory of lasers and masers makes use of models including damping [1], or in the study of decoherence phenomena [2]. Some authors have modeled dissipation by means of the theory of open systems or the thermal bath approach, in which a damped system is considered to be a subsystem of a bigger one with infinite degrees of freedom [3, 2]. However, damped systems are interesting in themselves as fundamental ones. In particular, the quantum damped harmonic oscillator, frequently described by the Caldirola-Kanai equation [4, 5], has attracted much attention, as it could be considered one of the simplest and paradigmatic examples of dissipative system.

The description of the quantum damped harmonic oscillator by the Caldirola-Kanai model, which includes a time-dependent Hamiltonian, has been considered to have some flaws. For instance, it is claimed that uncertainty relations are not preserved under time evolution and could eventually be violated [6, 7]. Many considerations were made in this direction. For example, Dekker in [8] introduced complex variables and noise operators to tackle the problem, claiming that no dynamical description in terms of a Schrödinger wave function can be expected to exist. In [9], a non-linear Schrödinger-Langevin wave equation was proposed as the starting

point in formulating the quantum theory. However, this inconsistency seems to be associated with a confusion between canonical momentum and "physical" momentum [10].

Despite these considerations about the Caldirola-Kanai model, many developments went ahead. Coherent states were calculated in [11] by finding creation and annihilation operators, built out of operators which commute with the Schrödinger equation. The corresponding number operator turns out to be an auxiliary, conserved operator, obviously different from the time-dependent Hamiltonian. This paper also defined the so-called loss energy states for the damped harmonic oscillator. The famous report by Dekker [12] provides a historical overview of some relevant results.

The analysis of damping from the symmetry point of view has proved to be especially fruitful. In a purely classical context, the symmetries of the equation of the damped harmonic oscillator with time-dependent parameters were found in [13]. Two comprehensive articles, [14, 15], are of special interest. In those papers the authors found, for the damped harmonic oscillator, finite-dimensional point symmetry groups for the corresponding Lagrangian (the un-extended Schrödinger group [16]) and the equations of motion ($SL(3,\mathbb{R})$) respectively, and an infinite contact one for the set of trajectories of the classical equation. They singled out a "non-conventional" Hamiltonian from those generators of the symmetry, recovering some results from [11]. Then, they concluded that the damped harmonic oscillator should not be claimed to be dissipative at all at the quantum level, since this "non-conventional" Hamiltonian is conserved and even related to an oscillator with variable frequency. In any case, it still remains to address the symmetry role of the time translation generator $i\hbar\frac{\partial}{\partial t}$. In fact, $i\hbar\frac{\partial}{\partial t}$ acting on a solution of the Schrödinger equation is no longer a solution. As a consequence, the time evolution operator $\hat{U}$ does not constitute a uniparametric group of unitary transformations. Equivalently, the solution of the equation $i\hbar\frac{\partial}{\partial t}\hat{U} = \hat{H}(t)\hat{U}$ is not $e^{\frac{i}{\hbar}t\hat{H}(t)}$, but rather the time ordered product $Te^{-\frac{i}{\hbar}\int \hat{H}(t)dt}$, referring to the Neuman series, or the Magnus series $e^{-\frac{i}{\hbar}\hat{\Omega}(t)}$ [17].

Many papers related to the Caldirola-Kanai model keep appearing, showing that the debate about fundamental quantum damping is far from being closed. We can mention [18], where the driven damped harmonic oscillator is analyzed, or the review [19]. Even the possible choices of classical Poisson structures and Hamiltonians, or generalizations to the non-commutative plane, have deserved attention as recently as in [20] and [21], respectively.

In fact, in [22], the authors provided a neat framework to study this model, based on a quantum generalization of the Arnold transformation [23]. The integrals of motion and symmetries were identified and exploited to calculate wave functions, basic operators and the exact time evolution operator.

Besides the Caldirola-Kanai model, the Bateman's dual system appears as an alternative description of dissipation in the damped harmonic oscillator. In his original paper [28], Bateman looked for a variational principle for equations of motion with a friction term linear in velocity, but he allowed the presence of extra equations. This trick effectively doubles the number of degrees of freedom, introducing a time-reversed version of the original damped harmonic oscillator, which acts as an energy reservoir and could be considered as an effective description of a thermal bath. The Hamiltonian that describes this system was rediscovered by Feschbach and Tikochinsky [29, 30, 31, 12] and the corresponding quantum theory was immediately analyzed.

Some issues regarding the Bateman's system arose. The Hamiltonian presents a set of complex eigenvalues of the energy (see [33] and references therein), and the vacuum of the theory decays with time. This last feature was treated in [32], where Celeghini et al. suggested that the quantum theory of the dual system could find a more natural framework in quantum field theory[1]. On the other hand, in [33] the generalized eigenvectors corresponding to the

[1]We feel that the ultimate reason is nevertheless the lack of a vacuum representation of the rele-

complex eigenvalues are interpreted as resonant states.

Bateman's dual system is still frequently discussed [34]. Many authors have considered this model as a *good* starting point for the formulation of the quantum theory of dissipation. One of the aims of this chapter will be to show that the study of the symmetries of the Caldirola-Kanai model leads to the Bateman's dual system, thus to be considered as a *natural* starting point for the study of quantum dissipation.

The purpose of this article is to throw some light on the subject of quantum dissipation by putting together all those bricks with the guide of symmetry. We begin in Section 2 by recalling the results from [22], in the case of the damped harmonic oscillator with constant coefficients. In particular, by using the quantum Arnold transformation, we import basic operators from the free particle system, which satisfy the condition of being integrals of the motion and close a Heisenberg algebra. Also, the complete set of symmetries of the quantum free particle, the Schrödinger group, can be realized on the Caldirola-Kanai model, providing as many conserved quantities as in the free particle.

Time translations in the non-free system do not belong the imported Schrödinger group. This is to be expected, as the classical equation of motion includes a friction term and the energy in this system is not conserved. The following question immediately arises: Is there any finite-dimensional group of symmetry containing time translations and, at least, the basic operators? The answer is 'yes', and Section 3 pays attention to this question in the case of the damped harmonic oscillator and the surprising consequences of the subsequent calculation: for this symmetry to act properly, it is necessary to enlarge the physical system with a new degree of freedom, corresponding to a new particle with interesting properties. This could be understood as a very simple version of the gauge principle, in which a bigger symmetry for the original "free" system is imposed. In fact, this new system with two degrees of freedom *is* the Bateman's dual system. Taking advantage of the symmetry approach, we will go a bit further and provide a group law corresponding to the symmetries of the dual system (Section 3.3).

With the light of this group law, in Section 4 we give an analysis of the quantization of the dual system that we have encountered. In particular, we show that it is possible to find a first-order Schrödinger equation (Subsection 4.1), from which the wave functions and the energy spectrum can be obtained, as well as the more usual second-order equation. In Subsection 4.2 we illustrate how the Caldirola-Kanai system can be recovered by means of a constraint process. An Appendix is devoted to the study of an infinite dimensional symmetry algebra for the damped particle.

2 Basic operators in the Caldirola-Kanai model

Let us first introduce the Caldirola-Kanai equation, which is a Schrödinger equation for the Damped Harmonic Oscillator (DHO):

$$i\hbar\frac{\partial\phi}{\partial t} = \hat{H}_{DHO}\,\phi \equiv -\frac{\hbar^2}{2m}e^{-\gamma t}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}m\omega^2x^2e^{\gamma t}\phi\,, \tag{1}$$

where γ and ω are constants defining the system. It is derived by standard canonical quantization from a time-dependent Hamiltonian whose quantum operator is $\hat{H}_{DHO}$. The corresponding classical equation of motion is

$$\ddot{x} + \gamma\dot{x} + \omega^2x = 0\,. \tag{2}$$

There are different ways to identify basic position and momentum operators associated with classical conserved quantities (Noether invariants). In general a conserved quantum operator

vant group (see Section 3).

$\hat{O}(t)$ must satisfy the relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}(t) \equiv \frac{\partial}{\partial t}\hat{O}(t) + \frac{i}{\hbar}[\hat{H}(t), \hat{O}(t)] = 0\,. \tag{3}$$

We find particularly interesting the Quantum Arnold Transformation (QAT) technique developed in [22].

The QAT (or, rather, the inverse) relates the Hilbert space $\mathcal{H}^G_\tau$ of solutions of the Schrödinger equation for the Galilean particle

$$i\hbar\frac{\partial\varphi}{\partial\tau} = -\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial\kappa^2}\,, \tag{4}$$

to the corresponding Hilbert space $\mathcal{H}_t$ of the DHO. The QAT can be written as

$$\begin{aligned} \hat{A}:\ \ \mathcal{H}_t \ \ &\longrightarrow \ \ \mathcal{H}^G_\tau \\ \phi(x,t) \longmapsto \ \varphi(\kappa,\tau) &= \hat{A}\big(\phi(x,t)\big) = A^*\big(\sqrt{u_2(t)}\, e^{-\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t)}\frac{\dot{u}_2(t)}{u_2(t)}x^2}\,\phi(x,t)\big)\,, \end{aligned} \tag{5}$$

where A^* denotes the pullback operation corresponding to the classical Arnold transformation A [23]:

$$\begin{aligned} A:\ \mathbb{R}\times T &\longrightarrow \mathbb{R}\times T' \\ (x,t) \longmapsto \ (\kappa,\tau) &= A\big((x,t)\big) = \left(\tfrac{x}{u_2}, \tfrac{u_1}{u_2}\right), \end{aligned} \tag{6}$$

and T and T' are open intervals of the real line containing $t = 0$ and $\tau = 0$, respectively. Here $u_1(t)$ and $u_2(t)$ are independent solutions of (2), satisfying the initial conditions $u_1(0) = 0, u_2(0) = 1, \dot{u}_1(0) = 1, \dot{u}_2(0) = 0$ and $W(t) \equiv \dot{u}_1(t)u_2(t) - u_1(t)\dot{u}_2(t)$ (see [22]). For the DHO they are:

$$u_1(t) = \frac{1}{\Omega}e^{-\frac{\gamma}{2}t}\sin\Omega t, \qquad u_2(t) = e^{-\frac{\gamma}{2}t}\cos\Omega t + \frac{\gamma}{2\Omega}e^{-\frac{\gamma}{2}t}\sin\Omega t, \tag{7}$$

for which $W(t) = e^{-\gamma t}$, and

$$\Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}\,. \tag{8}$$

Note that these solutions have good limit in the case of critical damping $\omega = \frac{\gamma}{2}$.

As already remarked, the basic symmetries of the free system are inherited by the DHO system, and we are now able to transform the infinitesimal generators of translations (the Galilean momentum operator $\hat{\pi}$, corresponding to the classical conserved quantity 'momentum') and non-relativistic boosts (the position operator $\hat{\kappa}$, corresponding to the classical conserved quantity 'initial position'). They are, explicitly,

$$\hat{\pi} = -i\hbar\frac{\partial}{\partial\kappa} \tag{9}$$

$$\hat{\kappa} = \kappa + \frac{i\hbar}{m}\tau\frac{\partial}{\partial\kappa}\,, \tag{10}$$

that is, those basic, canonically commuting operators with constant expectation values, that respect the solutions of the free Schrödinger equation, have constant matrix elements and fall down to well defined, time-independent operators in the Hilbert space of the free particle $L^2(\mathbb{R})$, $\mathcal{H}^G_{\tau=0}$.

The basic quantum operators, as derived by means of the inverse QAT on $\hat{\pi}$ and $\hat{\kappa}$ are:

$$\hat{P} = -i\hbar\frac{e^{-\frac{\gamma t}{2}}}{2\Omega}(2\Omega\,\cos\Omega t + \gamma\,\sin\Omega t)\frac{\partial}{\partial x} + m\frac{e^{\frac{\gamma t}{2}}}{4\Omega}\left(\gamma^2 + 4\Omega^2\right)\,\sin\Omega t\,x\,, \tag{11}$$

$$\hat{X} = \frac{e^{\frac{\gamma t}{2}}}{2\Omega}(2\Omega\,\cos\Omega t - \gamma\,\sin\Omega t)\,x + i\hbar\frac{e^{-\frac{\gamma t}{2}}}{m\Omega}\,\sin\Omega t\frac{\partial}{\partial x}\,. \tag{12}$$

with

$$\left[\hat{X}, \hat{P}\right] = i\hbar\,. \tag{13}$$

3 Deriving dissipative forces from a symmetry

Even though it is possible to set up a clear framework to deal with the quantum DHO system by employing the QAT, it does not provide by itself a well-defined operator associated with the actual time evolution. As mentioned in the Introduction, this is rooted in the fact that the conventional time evolution is not included in the symmetry group that can be imported from the free system: the Hamiltonian does not preserve the Hilbert space of solutions of the DHO Schrödinger equation. One may wonder what happens if time evolution symmetry is forced. We shall pursue this issue for the damped harmonic oscillator in this Section.

3.1 Time symmetry

Historically, Caldirola and Kanai derived their Hamiltonian from the Bateman one by means of time-dependent canonical transformations. Now we are going to proceed in the opposite direction, deriving Bateman Hamiltonian from Caldirola-Kanai one by purely symmetry considerations.

In the damped harmonic oscillator, neither the operator $i\hbar\frac{\partial}{\partial t}$, nor $\hat{H}_{DHO}$ (which coincides with the former on solutions) close under commutation with $\hat{X}$ and $\hat{P}$ (see equations (12) and (11)). We will impose the condition for the time translation to be a symmetry. But will do it in an elegant way, trying to close an algebra of (constant, symmetry generating) observables, taking advantage of the expressions for the basic operators obtained by the quantum Arnold transformation. So, we wonder if it is possible to incorporate $i\hbar\frac{\partial}{\partial t}$ into the basic Lie algebra of operators, trying to close an enlarged Lie algebra acting on the (possibly enlarged) Hilbert space $\mathcal{H}_t$. The answer to this question is in the affirmative, but it requires a delicate analysis. The resulting enlarged algebra will include $\hat{X}, \hat{P}, \hat{H} \equiv i\hbar\frac{\partial}{\partial t}$ and four more generators (plus the central one $\hat{I}$), denoted by $\hat{Q}, \hat{\Pi}, \hat{G}_1$ and $\hat{G}_2$[2].

Together with the generators $\hat{X}$ and $\hat{P}$ and the Hamiltonian, let us introduce the following operators:

$$\begin{aligned}
\hat{P} &= -i\hbar e^{-\frac{\gamma t}{2}}(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t)\frac{\partial}{\partial x} + m\,\frac{\omega^2}{\Omega}\,e^{\frac{\gamma t}{2}}\,\sin\Omega t\, x\,,\\
\hat{X} &= e^{\frac{\gamma t}{2}}(\cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t)\,x + i\hbar\frac{e^{-\frac{\gamma t}{2}}}{m\Omega}\,\sin\Omega t\frac{\partial}{\partial x}\\
\hat{\Pi} &= i\hbar e^{-\frac{\gamma t}{2}}\,(\cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t)\frac{\partial}{\partial x} - m\frac{\omega^2}{\Omega}\,e^{\frac{\gamma t}{2}}\,\sin\Omega t\, x\\
\hat{\hat{Q}} &= e^{\frac{\gamma t}{2}}(\cos\Omega t - \frac{3\gamma}{2\Omega}\sin\Omega t)\,x + i\hbar\frac{e^{-\frac{\gamma t}{2}}}{m\Omega}\,\sin\Omega t\frac{\partial}{\partial x}\\
\hat{G}_1 &= \frac{1}{4\Omega^2}\left(-4\omega^2 + \gamma^2\cos 2\Omega t + 2\gamma\Omega\sin 2\Omega t\right),\\
\hat{G}_2 &= -\frac{\gamma}{\Omega^2}\sin^2\Omega t\,,
\end{aligned}$$

[2]In the simpler case of the damped particle, infinitely many new generators can be included in its Lie algebra. See Appendix 4.2 for further details.

so that they close the seven-dimensional algebra:

$$
\begin{aligned}
&\left[\hat{X},\hat{P}\right] = i\hbar\hat{I} && \left[\hat{\tilde{Q}},\hat{\Pi}\right] = 2i\hbar\hat{G}_1 + i\hbar\hat{I} \\
&\left[\hat{X},\hat{\tilde{Q}}\right] = \frac{i\hbar}{m}\hat{G}_2 && \left[\hat{X},\hat{\Pi}\right] = i\hbar\hat{G}_1 \\
&\left[\hat{\tilde{Q}},\hat{P}\right] = -i\hbar\hat{G}_1 + i\hbar\gamma\hat{G}_2 && \left[\hat{P},\hat{\Pi}\right] = -i\hbar m\omega^2\hat{G}_2 \\
&\left[\hat{H},\hat{X}\right] = \frac{i\hbar}{m}\hat{\Pi} && \left[\hat{H},\hat{P}\right] = 2i\hbar m\omega^2\hat{X} - i\hbar m\omega^2\hat{\tilde{Q}} \\
&\left[\hat{H},\hat{\tilde{Q}}\right] = -2i\hbar\gamma\hat{X} - \frac{i\hbar}{m}\hat{P} + i\hbar\gamma\hat{\tilde{Q}} && \left[\hat{H},\hat{\Pi}\right] = -3i\hbar m\omega^2\hat{X} + 2i\hbar m\omega^2\hat{\tilde{Q}} - i\hbar\gamma\hat{\Pi} \\
&\left[\hat{H},\hat{G}_1\right] = -i\hbar\gamma\hat{G}_1 + 2i\hbar\omega^2\hat{G}_2 && \left[\hat{H},\hat{G}_2\right] = -2i\hbar\hat{G}_1 + i\hbar\gamma\hat{G}_2 - 2i\hbar\hat{I}
\end{aligned}
$$

We see that this algebra corresponds to a centrally extended algebra. The central extensions determine the actual basic conjugated pairs and classify possible quantizations. The operators $\hat{\tilde{Q}}$ and $\hat{\Pi}$ (plus $\hat{I}$) expand a Heisenberg-Weyl subalgebra, and $\hat{H}, \hat{G}_1$ and $\hat{G}_2$ expand a 2-D affine algebra (with $\hat{H}$ acting as dilations). However, in this realization $\hat{\tilde{Q}}$ and $\hat{\Pi}$ are not basic (this can be seen as an anomaly), and $\hat{H}$ and $\hat{G}_2$ are basic, resulting in time being a canonical variable. Clearly, this is not satisfactory, and an alternative description should be looked for.

Our strategy here is to consider other possible quantizations of the un-extended algebra. A detailed study of the (projective) representations of the enlarged (7+1) dimensional Lie algebra (that is, the possible central extensions) is going to show that there are three relevant kinds of representations, describing systems with different degrees of freedom.

Thinking of the algebra above as an abstract Lie algebra, it can be shown that a parameter k controls the central extensions which are allowed by the Jacobi identity:

$$
\begin{aligned}
&\left[\hat{X},\hat{P}\right] = i\hbar\hat{I} && \left[\hat{\tilde{Q}},\hat{\Pi}\right] = 2i\hbar\hat{G}_1 + i\hbar k\hat{I} \\
&\left[\hat{X},\hat{\tilde{Q}}\right] = \frac{i\hbar}{m}\hat{G}_2 && \left[\hat{X},\hat{\Pi}\right] = i\hbar\hat{G}_1 \\
&\left[\hat{\tilde{Q}},\hat{P}\right] = -i\hbar\hat{G}_1 + i\hbar\gamma\hat{G}_2 + i\hbar(1-k)\hat{I} && \left[\hat{P},\hat{\Pi}\right] = -i\hbar m\omega^2\hat{G}_2 \\
&\left[\hat{H},\hat{X}\right] = \frac{i\hbar}{m}\hat{\Pi} && \left[\hat{H},\hat{P}\right] = 2i\hbar m\omega^2\hat{X} - i\hbar m\omega^2\hat{\tilde{Q}} \\
&\left[\hat{H},\hat{\tilde{Q}}\right] = -2i\hbar\gamma\hat{X} - \frac{i\hbar}{m}\hat{P} + i\hbar\gamma\hat{\tilde{Q}} && \left[\hat{H},\hat{\Pi}\right] = -3i\hbar m\omega^2\hat{X} + 2i\hbar m\omega^2\hat{\tilde{Q}} - i\hbar\gamma\hat{\Pi} \\
&\left[\hat{H},\hat{G}_1\right] = -i\hbar\gamma\hat{G}_1 + 2i\hbar\omega^2\hat{G}_2 && \left[\hat{H},\hat{G}_2\right] = -2i\hbar\hat{G}_1 + i\hbar\gamma\hat{G}_2 - i\hbar(1+k)\hat{I}
\end{aligned}
$$

It is convenient to perform the shift:

$$\hat{Q} \equiv -\hat{\tilde{Q}} + (1-k)\hat{X} \, ,$$

so that the actual degrees of freedom diagonalize:

$$
\begin{aligned}
\left[\hat{X},\hat{P}\right] &= i\hbar\hat{I} & \left[\hat{Q},\hat{\Pi}\right] &= -i\hbar(k+1)\hat{G}_1 - i\hbar k\hat{I} \\
\left[\hat{X},\hat{Q}\right] &= -\frac{i\hbar}{m}\hat{G}_2 & \left[\hat{X},\hat{\Pi}\right] &= i\hbar\hat{G}_1 \\
\left[\hat{Q},\hat{P}\right] &= i\hbar\hat{G}_1 - i\hbar\gamma\hat{G}_2 & \left[\hat{P},\hat{\Pi}\right] &= -i\hbar m\omega^2\hat{G}_2 \\
\left[\hat{H},\hat{X}\right] &= \frac{i\hbar}{m}\hat{\Pi} & \left[\hat{H},\hat{P}\right] &= i\hbar m\omega^2(1+k)\hat{X} + i\hbar m\omega^2\hat{Q} \\
\left[\hat{H},\hat{Q}\right] &= i\hbar\gamma(1+k)\hat{X} + \frac{i\hbar}{m}\hat{P} & \left[\hat{H},\hat{\Pi}\right] &= -i\hbar m\omega^2(2k+1)\hat{X} \\
&\quad + i\hbar\gamma\hat{Q} + \frac{i\hbar}{m}(1-k)\hat{\Pi} & &\quad - 2i\hbar m\omega^2\hat{Q} - i\hbar\gamma\hat{\Pi} \\
\left[\hat{H},\hat{G}_1\right] &= -i\hbar\gamma\hat{G}_1 + 2i\hbar\omega^2\hat{G}_2 & \left[\hat{H},\hat{G}_2\right] &= -2i\hbar\hat{G}_1 + i\hbar\gamma\hat{G}_2 - i\hbar(1+k)\hat{I}
\end{aligned}
$$

We can see that the representations of this algebra include:

- For arbitrary k, a generic family with 3 degrees of freedom: $(\hat{X},\hat{P})$, $(\hat{Q},\hat{\Pi})$ and $(\hat{H},\hat{G}_2)$, then time being a canonical variable.
- For $k=1$, already described, an anomalous family with 2 degrees of freedom: $(\hat{X},\hat{P})$ and $(\hat{H},\hat{G}_2)$, then time being a canonical variable.
- For $k=-1$, a family with 2 degrees of freedom: $(\hat{X},\hat{P})$ and $(\hat{Q},\hat{\Pi})$.

Clearly, the interesting case is the third one, since it contains two degrees of freedom and time is not a canonical variable. Its algebra is given by:

$$
\begin{aligned}
\left[\hat{X},\hat{P}\right] &= i\hbar\hat{I} & \left[\hat{Q},\hat{\Pi}\right] &= i\hbar\hat{I} \\
\left[\hat{X},\hat{Q}\right] &= -\frac{i\hbar}{m}\hat{G}_2 & \left[\hat{X},\hat{\Pi}\right] &= i\hbar\hat{G}_1 \\
\left[\hat{Q},\hat{P}\right] &= i\hbar\hat{G}_1 - i\hbar\gamma\hat{G}_2 & \left[\hat{P},\hat{\Pi}\right] &= -i\hbar m\omega^2\hat{G}_2 \\
\left[\hat{H},\hat{X}\right] &= \frac{i\hbar}{m}\hat{\Pi} & \left[\hat{H},\hat{P}\right] &= i\hbar m\omega^2\hat{Q} \\
\left[\hat{H},\hat{Q}\right] &= \frac{i\hbar}{m}(\hat{P}+2\hat{\Pi}) & \left[\hat{H},\hat{\Pi}\right] &= i\hbar m\omega^2(\hat{X}-2\hat{Q}) \\
&\quad + i\hbar\gamma\hat{Q} & &\quad - i\hbar\gamma\hat{\Pi} \\
\left[\hat{H},\hat{G}_1\right] &= -i\hbar\gamma\hat{G}_1 + 2i\hbar\omega^2\hat{G}_2 & \left[\hat{H},\hat{G}_2\right] &= -2i\hbar\hat{G}_1 + i\hbar\gamma\hat{G}_2\,.
\end{aligned}
$$

In this case the operators $\hat{G}_1$ and $\hat{G}_2$ are gauge (they commute with the basic couples $(\hat{X},\hat{P})$ and $(\hat{Q},\hat{\Pi})$) and therefore are represented trivially.

The effective dimension of the algebra is $5+1$: $(\hat{X},\hat{P})$, $(\hat{Q},\hat{\Pi})$, $\hat{H}$ and $\hat{I}$.

$$
\begin{aligned}
\left[\hat{X},\hat{P}\right] &= i\hbar\hat{I} & \left[\hat{Q},\hat{\Pi}\right] &= i\hbar\hat{I} \\
\left[\hat{X},\hat{Q}\right] &= 0 & \left[\hat{X},\hat{\Pi}\right] &= 0 \\
\left[\hat{Q},\hat{P}\right] &= 0 & \left[\hat{P},\hat{\Pi}\right] &= 0 \\
\left[\hat{H},\hat{X}\right] &= \frac{i\hbar}{m}\hat{\Pi} & \left[\hat{H},\hat{P}\right] &= i\hbar m\omega^2\hat{Q} \\
\left[\hat{H},\hat{Q}\right] &= \frac{i\hbar}{m}(\hat{P}+2\hat{\Pi}) & \left[\hat{H},\hat{\Pi}\right] &= i\hbar m\omega^2(\hat{X}-2\hat{Q}) \\
&\quad + i\hbar\gamma\hat{Q} & &\quad - i\hbar\gamma\hat{\Pi}\,.
\end{aligned}
$$

Here $\hat{H}$ is not a basic operator, and can be written in terms of the basic ones in an irreducible representation:

$$\hat{H} = -\frac{1}{m}\hat{\Pi}\hat{P} - \frac{\gamma}{2}(\hat{Q}\hat{\Pi}+\hat{\Pi}\hat{Q}) - \frac{\hat{\Pi}^2}{m} + m\omega^2\hat{X}\hat{Q} - m\omega^2\hat{Q}^2\,.$$

The classical version of the Hamiltonian is:

$$H = -\frac{1}{m}\Pi P - \gamma Q\Pi - \frac{\Pi^2}{m} + m\omega^2 XQ - m\omega^2 Q^2\,. \tag{14}$$

3.2 Bateman's system

The classical Hamiltonian (14) can be transformed, using the linear, constant, canonical transformation:

$$
\begin{aligned}
X &= \frac{m\omega^2 y - (p_y + m\frac{\gamma}{2}x)i\Omega}{m\omega\sqrt{-\gamma i\Omega}} \\
P &= \frac{\omega(p_x - m\frac{\gamma}{2}y + mxi\Omega)}{\sqrt{-\gamma i\Omega}} \\
Q &= \frac{m\omega^2 y - (p_y - m\frac{\gamma}{2}x)i\Omega}{m\omega\sqrt{-\gamma i\Omega}} \\
\Pi &= -\frac{\omega(p_x + m\frac{\gamma}{2}y + mxi\Omega)}{\sqrt{-\gamma i\Omega}}\,,
\end{aligned} \tag{15}
$$

into the Bateman dual Hamiltonian

$$H_B = \frac{p_x p_y}{m} + \frac{\gamma}{2}(yp_y - xp_x) + m\Omega^2 xy\,, \tag{16}$$

that describes a damped particle (x, p_x) and its time reversal (y, p_y):

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0\,, \qquad \ddot{y} - \gamma\dot{y} + \omega^2 y = 0\,. \tag{17}$$

The quantum Bateman Hamiltonian is:

$$\hat{H}_B = \frac{\hat{p}_x\hat{p}_y}{m} + \frac{\gamma}{2}(\hat{y}\hat{p}_y - \hat{x}\hat{p}_x) + m\Omega^2\hat{x}\hat{y}\,, \tag{18}$$

and the Schrödinger equation for the Bateman's system is given by[3]:

$$i\hbar\frac{\partial\phi(x,y,t)}{\partial t} = \left[-\frac{\hbar^2}{m}\frac{\partial^2}{\partial x\partial y} - i\hbar\frac{\gamma}{2}(y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}) + m\Omega^2 xy\right]\phi(x,y,t)\,. \tag{19}$$

The system is conservative, so our objective of including time evolution among the symmetries has been accomplished, at the cost of including a new degree of freedom. $\hat{H}_B$ closes a 5+1 dimensional algebra with $(\hat{x},\hat{p}_x)$ and $(\hat{y},\hat{p}_y)$:

$$\begin{aligned}
[\hat{x},\hat{p}_x] &= i\hbar\hat{I} & [\hat{y},\hat{p}_y] &= i\hbar\hat{I}\\
[\hat{x},\hat{y}] &= 0 & [\hat{x},\hat{p}_y] &= 0\\
[\hat{y},\hat{p}_x] &= 0 & [\hat{p}_x,\hat{p}_y] &= 0\\
\left[\hat{H}_B,\hat{x}\right] &= \frac{i\hbar}{m}(-\hat{p}_y + m\frac{\gamma}{2}\hat{x}) & \left[\hat{H}_B,\hat{p}_x\right] &= i\hbar(-\frac{\gamma}{2}\hat{p}_x + m\Omega^2\hat{y})\\
\left[\hat{H}_B,\hat{y}\right] &= \frac{i\hbar}{m}(-\hat{p}_x - m\frac{\gamma}{2}\hat{y}) & \left[\hat{H}_B,\hat{p}_y\right] &= i\hbar(\frac{\gamma}{2}\hat{p}_y + m\Omega^2\hat{x})\,.
\end{aligned} \tag{20}$$

However, it has been argued that the quantum Bateman's system possesses inconsistencies, like complex eigenvalues and non-normalizable eigenstates. Chruściński & Jurkowski [33] showed that $\hat{H}_B$ has real, continuous spectrum (we will provide a prove of this in Subsection 4.1), and that the complex eigenvalues are associated with resonances, which in last instance are the responsible of dissipation.

3.3 Bateman's group law

The Lie algebra (20) can be exponentiated to give a Lie group, whose group law we have found to be:

$$\begin{aligned}
t'' &= t' + t\\
x'' &= x + x'\,e^{-\frac{\gamma t}{2}}\,\cos\Omega t + \frac{p'_y}{m\Omega}\,e^{-\frac{\gamma t}{2}}\,\sin\Omega t\\
y'' &= y + y'\,e^{\frac{\gamma t}{2}}\,\cos\Omega t + \frac{p'_x}{m\Omega}\,e^{\frac{\gamma t}{2}}\,\sin\Omega t\\
p''_x &= p_x + p'_x\,e^{\frac{\gamma t}{2}}\,\cos\Omega t - m\Omega\,y'\,e^{\frac{\gamma t}{2}}\,\sin\Omega t\\
p''_y &= p_y + p'_y\,e^{-\frac{\gamma t}{2}}\,\cos\Omega t - m\Omega\,x'\,e^{-\frac{\gamma t}{2}}\,\sin\Omega t\\
\zeta'' &= \zeta'\zeta e^{\frac{i}{\hbar}\{y' p_y\,e^{\frac{\gamma t}{2}}\,\cos\Omega t - x\,p'_x\,e^{\frac{\gamma t}{2}}\,\cos\Omega t + m\Omega\,x\,y'\,e^{\frac{\gamma t}{2}}\,\sin\Omega t + \frac{1}{m\Omega}\,p'_x p_y\,e^{\frac{\gamma t}{2}}\,\sin\Omega t\}}\,.
\end{aligned}$$

This group law had not been considered previously in the literature, up to the author's knowledge.

[3]The Bateman system admits an equivalent description in terms of a real, first-order, Schrödinger equation (see Subsection 4.1)

The corresponding left-invariant vector fields can be computed:

$$\begin{aligned}
\tilde{X}^L_t &= \frac{\partial}{\partial t} + (-\frac{\gamma}{2}x + \frac{p_y}{m})\frac{\partial}{\partial x} + (\frac{\gamma}{2}y + \frac{p_x}{m})\frac{\partial}{\partial y} \\
&\qquad\qquad + (\frac{\gamma}{2}p_x - m\Omega^2 y)\frac{\partial}{\partial p_x} + (-\frac{\gamma}{2}p_y - m\Omega^2 x)\frac{\partial}{\partial p_y} \\
\tilde{X}^L_x &= \frac{\partial}{\partial x} - \frac{p_x}{\hbar}\,\Xi \\
\tilde{X}^L_y &= \frac{\partial}{\partial y} \\
\tilde{X}^L_{p_x} &= \frac{\partial}{\partial p_x} \\
\tilde{X}^L_{p_y} &= \frac{\partial}{\partial x} + \frac{y}{\hbar}\,\Xi\,,
\end{aligned}$$

and also the right-invariant ones:

$$\begin{aligned}
\tilde{X}^R_t &= \frac{\partial}{\partial t} \\
\tilde{X}^R_x &= e^{-\frac{\gamma t}{2}}\,\cos\Omega t\frac{\partial}{\partial x} - m\Omega\, e^{-\frac{\gamma t}{2}}\,\sin\Omega t\frac{\partial}{\partial p_y} \\
\tilde{X}^R_y &= e^{\frac{\gamma t}{2}}\,\cos\Omega t\frac{\partial}{\partial y} - m\Omega\, e^{\frac{\gamma t}{2}}\,\sin\Omega t\frac{\partial}{\partial p_x} + \frac{1}{\hbar}(p_y\, e^{\frac{\gamma t}{2}}\,\cos\Omega t + m\Omega\, x\, e^{\frac{\gamma t}{2}}\,\sin\Omega t)\,\Xi \\
\tilde{X}^R_{p_x} &= e^{\frac{\gamma t}{2}}\,\cos\Omega t\frac{\partial}{\partial p_x} + \frac{1}{m\Omega}\,e^{\frac{\gamma t}{2}}\,\sin\Omega t\frac{\partial}{\partial y} - \frac{1}{\hbar}(x\, e^{\frac{\gamma t}{2}}\,\cos\Omega t - \frac{1}{m\Omega}\,p_y\, e^{\frac{\gamma t}{2}}\,\sin\Omega t)\,\Xi \\
\tilde{X}^R_{p_y} &= e^{-\frac{\gamma t}{2}}\,\cos\Omega t\frac{\partial}{\partial p_y} + \frac{1}{m\Omega}\,e^{-\frac{\gamma t}{2}}\,\sin\Omega t\frac{\partial}{\partial x}\,.
\end{aligned}$$

These vector fields close the Lie algebra (20), provided that obvious identifications are made.

4 A note on the quantization of the Bateman's dual system

4.1 First-order Schrödinger equation

Usual Canonical Quantization leads to either position space or momentum space representation and a corresponding second-order Schödinger equation. However, inspecting the Bateman's Lie algebra, it is possible to check that a full first-order polarization exists:

$$\mathcal{P} = \langle \tilde{X}^L_y, \tilde{X}^L_{p_x}, \tilde{X}^L_t \rangle\,.$$

The first two polarization conditions determine that wave functions are ($U(1)$-functions) depending only on (x, p_y, t). The last polarization equation $\tilde{X}^L_t\psi = 0$ determines the condition on functions on the reduced space, $\phi(x, p_y, t)$:

$$\frac{\partial\phi}{\partial t} = -(-\frac{\gamma}{2}x + \frac{p_y}{m})\frac{\partial\phi}{\partial x} - (-\frac{\gamma}{2}p_y - m\Omega^2 x)\frac{\partial\phi}{\partial p_y}\,.$$

We have arrived at a first-order partial differential equation that must be interpreted as a first-order Schrödinger equation in a mixed representation position-momentum. In fact, the same result can be obtained performing Canonical Quantization for the Bateman Hamiltonian $\hat{H}_B$ in this mixed representation. Let us emphasize that this has been strongly suggested by the group structure and the GAQ algorithm.

The corresponding time-independent Schrödinger equation is written:

$$(\frac{\gamma}{2}x - \frac{p_y}{m})\frac{\partial\phi}{\partial x} + (\frac{\gamma}{2}p_y + m\Omega^2 x)\frac{\partial\phi}{\partial p_y} = E\phi\,.$$

The general solution of this equation can be found in terms of the complex variable $z \equiv p_y + im\Omega x$:

$$\phi(z) = \Big(\frac{z}{z^*}\Big)^{\frac{E}{2\hbar\Omega}} f\big(zz^*\Big(\frac{z}{z^*}\Big)^{\frac{i\gamma}{2\Omega}}\big)\,,$$

where f is an arbitrary function of its argument.

Let us focus in the case of underdamping, where Ω is real. We must determine whether ϕ is well defined. To this end, we assume that f is a power of its argument, $(zz^*)^{\tilde\lambda}\Big(\frac{z}{z^*}\Big)^{\frac{i\gamma\tilde\lambda}{2\Omega}}$, and we write:

$$\phi(z) = \Big(\frac{z}{z^*}\Big)^{\frac{E}{2\hbar\Omega}}(zz^*)^{\tilde\lambda}\Big(\frac{z}{z^*}\Big)^{\frac{i\gamma\tilde\lambda}{2\Omega}} = \Big(\frac{z}{z^*}\Big)^{\frac{E+i\hbar\tilde\lambda\gamma}{2\hbar\Omega}}(zz^*)^{\tilde\lambda}\,.$$

Now, the wave function have to be well-defined. This imposes a "quantization" condition on the spectrum.

On the one hand, recall that zz^* is real. For ϕ to be at least Dirac-delta normalizable, $\tilde\lambda$ must be chosen to be pure imaginary:

$$\tilde\lambda = i\lambda\,, \qquad \lambda \in \mathbb{R}\,.$$

On the other hand, $\frac{z}{z^*}$ is a pure phase, with twice the argument of z. The exponent of $\frac{z}{z^*}$ must be half-integer so that we can get a well-defined function of z:

$$E - \hbar\gamma\lambda = n\hbar\Omega \quad \Rightarrow \quad \boxed{E = n\hbar\Omega + \lambda\hbar\gamma}\,.$$

That is, we obtain a spectrum which has an integer part and a continuous part.

These results coincide with those in [33], although here they are obtained in a quicker and neater way. The reason is that they quantize angular variables and hence the basic operator "multiply by the angle" is not defined. We have avoided this problem. However, it is somewhat surprising that both spectrums coincide.

The fact that the spectrum of $\hat H_B$ has an integer part and a continuous part suggests that $\hat H_B$ can be split into a compact operator (of the harmonic oscillator type) and another operator with an unbounded, continuous spectrum. This splitting should be found in the Lie algebra of the Bateman's group, and is under investigation.

4.2 Back to Caldirola-Kanai system

Historically, Bateman firstly derived H_B, and later Caldirola and Kanai obtained H_{DHO} using time-dependent canonical transformations. Here we have gone the opposite way, started from H_{DHO} and derived H_B closing a finite Lie algebra. Now we wonder if we can do the way back to the Caldirola-Kanai system. The answer, again, is positive, and can be achieved by using constraints. To know how to proceed, let us analyse first the classical case.

Classically, Bateman's system and a pair of dual Caldirola-Kanai systems share the same second-order equations of motion. If we impose them to share the first-order, Hamilton equations, the following constraint must be satisfied:

$$\begin{aligned} y &= \frac{\omega^2}{\Omega^2}e^{\gamma t}x + \frac{\gamma}{2m\Omega^2}p_x \\ p_y &= e^{\gamma t}p_x + m\frac{\gamma}{2}x\,. \end{aligned} \tag{21}$$

These constraints, although time dependent, preserve the equations of motion since they are equivalent to a relation among initials constants:

$$
\begin{aligned}
y_0 &= \frac{\omega^2}{\Omega^2}x_0 + \frac{\gamma}{2m\Omega^2}p_{x0} \\
p_{y0} &= p_{x0} + m\frac{\gamma}{2}x_0 \,.
\end{aligned}
\tag{22}
$$

These constraints can be seen to be of second-order type, besides being time-dependent, therefore care should be taken when imposing them: Dirac theory for constraints can be used or we can embed the constraints in a time-dependent canonical transformation before applying them.

But we are interested in the quantum derivation. Therefore we try to impose the operator constraints:

$$
\begin{aligned}
\hat{y} - \frac{\omega^2}{\Omega^2}\hat{x} - \frac{\gamma}{2m\Omega^2}\hat{p}_x &= 0 \\
\hat{p}_y - \hat{p}_x - m\frac{\gamma}{2}\hat{x} &= 0 \,,
\end{aligned}
\tag{23}
$$

but only one of them can be imposed, since the operators at the lhs of the equations canonically commute: they are of second order type. At the quantum level, only one of them can be imposed, therefore we must select one of them. If we impose the constraint,

$$
\hat{y} = \frac{\omega^2}{\Omega^2}\hat{x} + \frac{\gamma}{2m\Omega^2}\hat{p}_x \,, \tag{24}
$$

the Hilbert space reduces to those functions verifying:

$$
\phi(x,y,t) = e^{\frac{i e^{-\gamma t} m\Omega y \mathrm{Csc}^2(\Omega t)\left(\gamma\Omega y \mathrm{Cos}(2\Omega t) + 2\left(\omega^2 e^{\gamma t} x' - \Omega^2 y\right)\mathrm{Sin}(2\Omega t)\right)}{4\hbar\omega^2}} \psi(x',t) \,, \tag{25}
$$

where $x' = x + \frac{\Omega^2}{2\omega^2} y e^{-\gamma t}\mu(t)$, and $\mu(t) = (2 - \frac{\gamma}{\Omega}\mathrm{Cot}(\Omega t))$. The Schrödinger equation for the Bateman's system reduces to:

$$
i\hbar\frac{\partial\psi(x',t)}{\partial t} = \left[-\frac{\Omega^2\hbar^2}{2m\omega^2}e^{-\gamma t}\mu(t)\frac{\partial^2}{\partial x'^2} - \frac{1}{2}i\hbar x'\Omega\mu(t)\frac{\partial}{\partial x'} + i\hbar\frac{\Omega^2}{\gamma}(\mu(t) - 2)\right]\psi(x',t) \,. \tag{26}
$$

When constraints are imposed, not all the operators acting on the original Hilbert space preserve the constrained Hilbert space. The notion of "good" (usually denoted gauge-independent in constrained gauge theories) operators as those preserving the constrained Hilbert space naturally emerges.

In most of the cases "good" operators are characterized as those commuting with the constraints (see [40] for a detailed account of quantum constraints in a group-theoretical setting and a more general characterization of "good" operators). In this case they are:

$$
\hat{p}_x + \frac{2m\omega^2}{\gamma}\hat{x} \qquad \hat{p}_y - \frac{2m\Omega^2}{\gamma}\hat{x} \,. \tag{27}
$$

Note that $\hat{H}_B$ (nor $i\hbar\frac{\partial}{\partial t}$) is not among the "good" operators since it does not preserve the constrained Hilbert space. Therefore, time invariance is lost in the process of going from the Bateman's system to the Caldirola-Kanai system due to the very nature of the constraints imposed.

Now let us perform the transformation

$$\psi(x',t) = e^{-i\frac{m\omega^2}{\hbar\Omega}x'^2 f(t)} g(t)\chi(\kappa,\tau)\,, \tag{28}$$

where

$$f(t) = -\frac{e^{\gamma t}}{4\Omega\mu(t)^2\tau'(t)}\left((-\gamma(2+\mathrm{Cos}(2\Omega t)) + 2\Omega\mathrm{Sin}(2\Omega t))\tau'(t) - \gamma\mu(t)\tau'(t)^2 + \mu(t)\tau''(t)\right) \tag{29}$$

$$g(t) = e^{-\frac{1}{4}\gamma\tau}\left(-\frac{\tau'(t)}{\Omega\mathrm{Sin}^2(\Omega t)\mu(t)}\right)^{1/4} \tag{30}$$

$$\kappa = x' e^{\frac{\gamma}{2}(t-\tau)}\frac{\omega}{\Omega}\sqrt{\frac{\tau'(t)}{\mu(t)}} \tag{31}$$

$$\tau(t) = \frac{1}{\Omega}\mathrm{ArcTan}\left[\frac{A\frac{\gamma^2}{\Omega^2}}{\mu(t)^2}\right]\,,\ A \in \mathbb{R} - \{0\}\,. \tag{32}$$

The Schrödinger equation finally transforms into:

$$i\hbar\frac{\partial}{\partial\tau}\chi(\kappa,\tau) = \left[-\frac{\hbar^2}{2m}e^{-\gamma\tau}\frac{\partial^2}{\partial\kappa^2} + \frac{1}{2}m\omega^2\kappa^2 e^{\gamma\tau}\right]\chi(\kappa,\tau)\,, \tag{33}$$

which is the Caldirola-Kanai equation in the variables (κ,τ). Even more, the two independent operators (27) preserving the constrained Hilbert space turn, under the previous transformation, to the basic operators for the Caldirola-Kanai system $\hat{x}(t)$ and $\hat{p}(t)$. Therefore, we have recovered completely the Caldirola-Kanai system from the Bateman's system by imposing one constraint.

It should be stressed that $\tau'(0) = 0$, therefore the time transformation is singular at the origin and there are two disconnected regions, one with $t > 0$ and other with $t < 0$. It also turns out that $sign(\tau) = sign(A)$, therefore choosing appropriately the sign of A in each case we can map $t > 0$ to $\tau > 0$ and $t < 0$ to $\tau < 0$, respectively.

This kind of behavior coincides with the results of other authors (see [33]) where, starting with the Bateman's system, they obtain two subspaces $\mathcal{S}^{\pm}$ for which the restriction of the one parameter group of unitary time-evolution operators $\hat{U}(t) = e^{-\frac{i}{\hbar}t\hat{H}_B}$ produces two semigroups of operators, for $t < 0$ and $t > 0$.

Therefore, starting from the quantum, conservative, Bateman's system we have arrived to the quantum, time-dependent, Caldirola-Kanai system. All the process we have performed can be schematically showed as:

$$\begin{array}{ccc} \text{Bateman} & \overset{\text{Constraint}}{\Longrightarrow} & \text{Caldirola-Kanai} \\ t \in \mathbb{R} & & t \in \mathbb{R}^+ \text{ or } t \in \mathbb{R}^- \\ \text{Conservative} & \underset{\text{Closing algebra}}{\Longleftarrow} & \text{Dissipative} \end{array} \tag{34}$$

Appendix: Infinite-dimensional symmetry in the damped particle

In this appendix, we turn our attention to the damped particle as the simplest case of physical system subjected to a dissipative force and perform a similar analysis to that carried out in Subsection 3.1 for the damped harmonic oscillator, forcing the introduction of the time symmetry.

The basic operators for the damped particle (obtained as the $\omega \to 0$ of the damped harmonic oscillator, see eq. (11-12))

$$\hat{P} = -i\hbar\frac{\partial}{\partial x}, \quad \hat{X} = x + \frac{i\hbar}{m\gamma}(1 - e^{-\gamma t})\frac{\partial}{\partial x}\,. \tag{35}$$

Reminding the operators (35), and renaming $\hat{P} \equiv \hat{P}_0$, we introduce operators $\hat{P}_n$ and $\hat{Y}_n$ (n an integer)

$$\hat{H}_G = i\hbar e^{\gamma t}\frac{\partial}{\partial t} \qquad \hat{H}_{DP} = i\hbar\frac{\partial}{\partial t} \tag{36}$$

$$\hat{P}_n = -i\hbar e^{-\gamma n t}\frac{\partial}{\partial x} \qquad \hat{Y}_n = i e^{-\gamma n t} \tag{37}$$

$$\hat{X} = x + \frac{i\hbar}{m\gamma}(1 - e^{\gamma t})\frac{\partial}{\partial x}\,. \tag{38}$$

It is interesting that they close an infinite-dimensional Lie algebra

$$\begin{aligned}
[\hat{H}_G, \hat{P}_n] &= -i\hbar\gamma n\hat{P}_{n-1} & [\hat{H}_{DP}, \hat{P}_n] &= -i\hbar\gamma n\hat{P}_n \\
[\hat{H}_G, \hat{X}] &= -i\frac{\hbar}{m}\hat{P}_0 & [\hat{H}_{DP}, \hat{X}] &= -i\frac{\hbar}{m}\hat{P}_1 \\
[\hat{H}_G, \hat{H}_{DP}] &= -i\hbar\gamma\hat{H}_G & [\hat{H}_{DP}, \hat{Y}_n] &= -i\hbar\gamma n\hat{Y}_n \\
[\hat{H}_G, \hat{Y}_n] &= -i\hbar\gamma n\hat{Y}_{n-1} & [\hat{X}, \hat{P}_n] &= \hbar\hat{Y}_n
\end{aligned}$$

(other commutators vanish) which has the (centrally extended) Galilei algebra as a subalgebra, noting that $\hat{Y}_0 = i$ is the central generator.

The generators in the right column, with $n = 0, 1$, also close a finite dimensional subalgebra in which $\hat{H}_{DP}$ is dynamical, conjugate of a combination of $\hat{P}_1$ and $\hat{Y}_1$, together with the couple $\hat{X}$, $\hat{P}_0$. A similar analysis to that of the damped harmonic oscillator is then possible, recovering the corresponding results when $\omega = 0$.

However, we have found that it is possible to enlarge the algebra to an infinite-dimensional one, at least in the case of the damped particle, and new degrees of freedom arise. A deeper analysis of this matter is under study.

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